2.4 nonhyperbolic equilibria and cobwebbing

Wednesday, January 27, 2021 3:00 PM

Recall: If $|f'(\bar{x})|>1$, then unstable $\qquad (x_{t+1}=f(x_t))$

$|f'(\bar{x})|<1$, then locally asymptotically stable.

Hyperbolic case where $|f'(\bar{x})|\neq 1$.

Let's take a look at the case when $f'(\bar{x})=1$.

(not going to analyze case when $f'(\bar{x})=-1$)

Thm 2.3 Suppose $f'(\bar{x})=1$, where $\bar{x}$ is an equilibrium pt of $x_{t+1}=f(x_t)$ and f''' continuous on an open interval $I\ni\bar{x}$.

(i) If $f''(\bar{x})\neq 0$, then $\bar{x}$ is unstable.

(ii) If $f''(\bar{x})=0$, and $f'''(\bar{x})>0$, then $\bar{x}$ is unstable

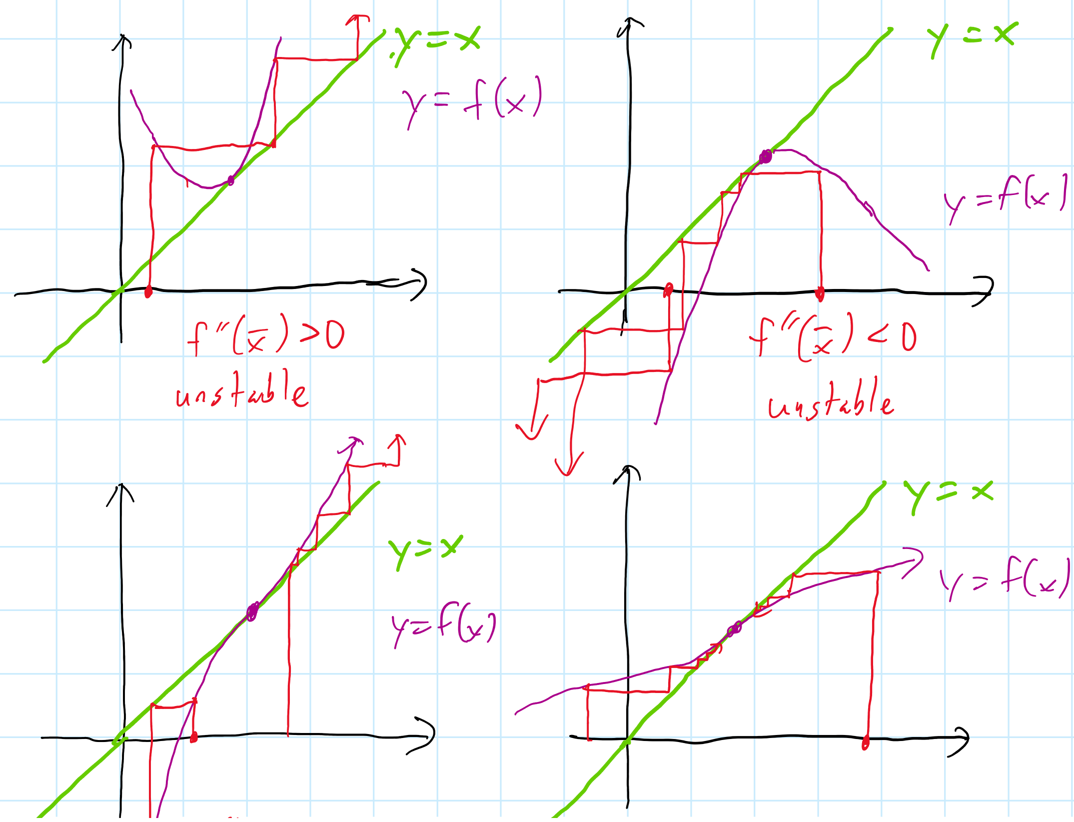
(iii) If $f''(\bar{x})=0$, and $f'''(\bar{x})<0$, then $\bar{x}$ is locally asymp. stable

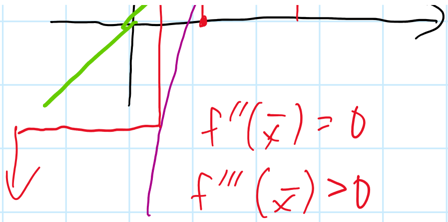
Note: We still can't say anything if $f'(\bar{x})=-1$

or if $f'(\bar{x})=1$, $f''(\bar{x})=0$, and $f'''(\bar{x})=0$.

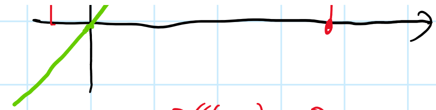
proof sketch:
(using cobwebbing)

$f'(\bar{x})=1$





$f''(\bar{x}) = 0$
$f'''(\bar{x}) > 0$
unstable



$f''(\bar{x}) = 0$
$f'''(\bar{x}) < 0$
locally asymptotically stable

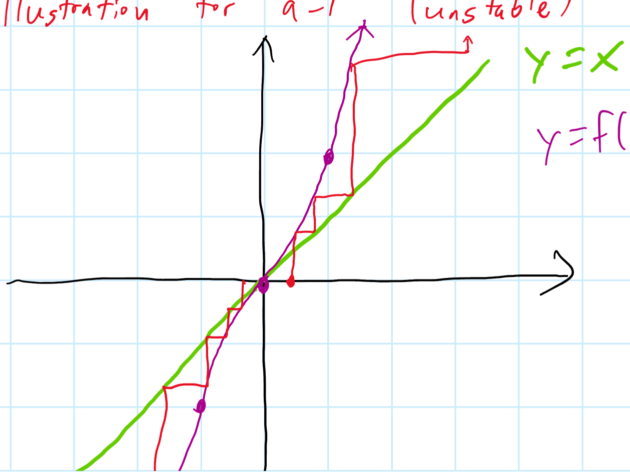
Exercise 2.5a $\quad x_{t+1} = a x_t^3 + x_t, \quad a \neq 0.$

Equilibria:
$$x = ax^3 + x$$
$$ax^3 = 0$$
$x = 0$ is the only real equilibria

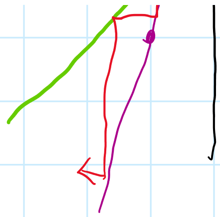
$$f(x) = ax^3 + x$$
$$f'(x) = 3ax^2 + 1 \implies f'(0) = 1$$
Need more derivatives
$$f''(x) = 6ax \implies f''(0) = 0$$
Still need more derivatives
$$f'''(x) = 6a \implies f'''(0) = 6a$$

If $a > 0$, $f'''(0) > 0$, so 0 is an unstable equilibrium
If $a < 0$, $f'''(0) < 0$, so 0 is a locally asymptotically stable equilibrium

Cobwebbing illustration for $a = 1$ (unstable)



$f(0) = 0$
$f(1) = 2 \qquad f(-1) = -2$
$f(2) = 10 \qquad f(-2) = -10$



More cobwebbing examples

Ex. 2.4 $\quad x_{t+1} = r - x_t^2 \quad , \quad f(x) = r - x^2$

Recall this has two equilibria : $\quad \bar{x}_\pm = \dfrac{-1 \pm \sqrt{1+4r}}{2}$

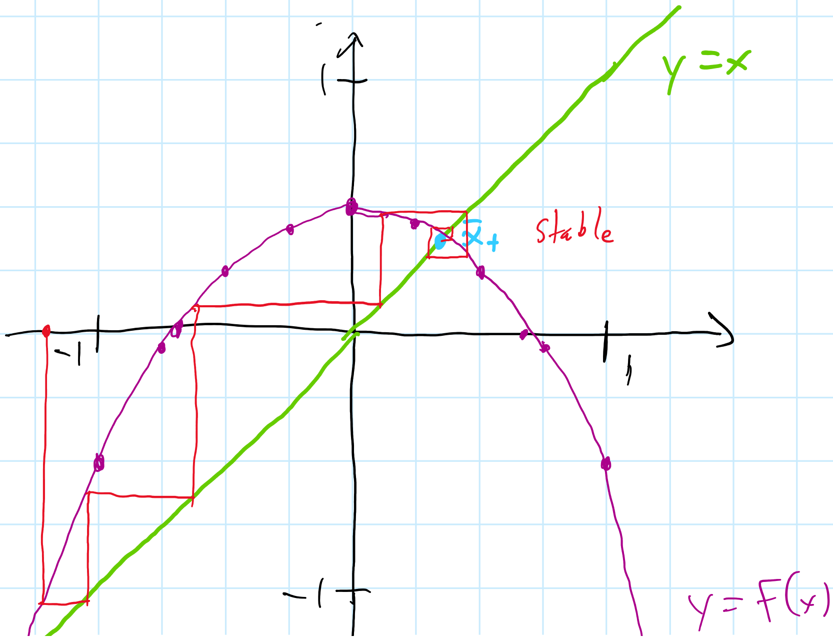
$\bar{x}_- = \dfrac{-1-\sqrt{1+4r}}{2}$ is unstable since $f'(x) = -2x$, $f'(\bar{x}_-) > 1$

$\bar{x}_+ = \dfrac{-1+\sqrt{1+4r}}{2}$ is locally asymp. stable if $r < \frac{3}{4}$
unstable if $r > \frac{3}{4}$.

Let's explicitly plot this out via cobwebbing when $r = \frac{1}{2}$

$\bar{x}_+ = \dfrac{-1+\sqrt{3}}{2} \approx 0.366$

$\bar{x}_- = \dfrac{-1-\sqrt{3}}{2} \approx -1.366$



$f(x) = \frac{1}{2} - x^2$

$f(0) = \frac{1}{2}$

$f(\frac{1}{4}) = \frac{7}{16}$

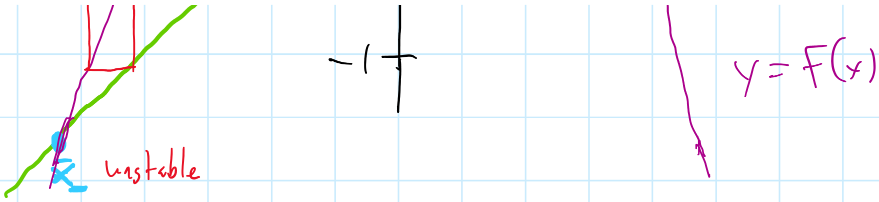
$f(\frac{1}{2}) = \frac{1}{4}$

$f(\frac{3}{4}) = -\frac{1}{16}$

$f(1) = -\frac{1}{2}$

$f(\frac{1}{\sqrt{2}}) = 0$

$\hookrightarrow 0.707$



Cobwebbing and periodic solutions

Recall: $x_{t+1} = r - x^2$ also has a 2-cycle which is stable iff $\frac{3}{4} < r < \frac{5}{4}$.

We can see this by cobwebbing

(1) Ordinary cobwebbing by plotting $y=x$ and $y=f(x)$ might show an alternating between the two.

(2) Cobwebbing of $y=x$ and $y=f(f(x))$

Recall: The 2-cycle is at $\bar{x}_{1,2} = \dfrac{1 \pm \sqrt{4r-3}}{2}$

Let $r=0$. Then $\bar{x}_{1,2} = 0, 1$.

Let's go to Python